APPLICATION OF INTEGRAL EQUATIONS TO HEAT CONDUCTION PROBLEMS IN WHICH THE HEAT TRANSFER COEFFICIENT VARIES

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The problem of heat conduction with a variable heat transfer coefficient is reduced to the solution of a Volterra integral equation of the second kind with a kernel having a singularity.

The temperature field of solids which have a heat transfer coefficient that varies with time is often encountered in practical problems. For example, there is the well-known problem of determining the temperature field of solids enveloped by pulsating flows of liquid or gas, which find application in problems involving the measurement of temperature in internal combustion engines [1, 2], in turbulence phenomena [3], etc.

There are a large number of papers devoted to this problem. In some of them, for example in [3-4], the problem of finding the temperature field is solved using various approximate methods. In [5], the case is considered when the relation Bi(Fo) can be represented in the form of a function which has a rational function by a Laplace transform.

In this paper we describe a general method of solving the problem of nonstationary heat conduction with a variable heat transfer coefficient, the dependence of which on time is arbitrary.

Consider, for example, the heating of an infinite plate of thickness δ with a variable heat transfer coefficient. In this case the mathematical formulation of the problem has the form

$$\frac{\partial T}{\partial \mathrm{Fo}} = \frac{\partial^2 T}{\partial \xi^2}, \tag{1}$$

$$T(\xi, 0) = T_0, \tag{2}$$

$$\frac{\partial T(0, \mathrm{Fo})}{\partial \xi} = 0, \tag{3}$$

$$\frac{\partial T(1, \mathrm{Fo})}{\partial \xi} = \mathrm{Bi}(\mathrm{Fo})[T_c(\mathrm{Fo}) - T(1, \mathrm{Fo})]. \tag{4}$$

Following [5], we will denote the right side of Eq. (4) by q(Fo). We will assume for the moment that the quantity q(Fo) is known. Problem (1)-(4) then reduces to a new problem with boundary conditions of the second kind

$$\frac{\partial T}{\partial \mathrm{Fo}} = \frac{\partial^2 T}{\partial \xi^2}, \tag{5}$$

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$$T(\xi, 0) = T_0, \tag{6}$$

$$\frac{\partial T(0, \mathrm{Fo})}{\partial \xi} = 0, \tag{7}$$

$$\frac{\partial T(1, \mathrm{Fo})}{\partial \xi} = q(\mathrm{Fo}). \tag{8}$$

Solving Eqs. (5)-(8) by the method of finite integral transformations [6], we obtain the solution for the temperature in the form

$$T(\xi, \mathrm{Fo}) = T_0 + \int_0^{\mathrm{Fo}} q(\omega)\, d\omega + 2\sum_{n=1}^{\infty}\left[(-1)^n \exp(-\mu_n^2 \mathrm{Fo}) \int_0^{\mathrm{Fo}} \exp(\mu_n^2 \omega)\, q(\omega)\, d\omega\right] \cos\mu_n \xi, \tag{9}$$

where $\mu_n = n\pi$.

Assuming $\xi = 1$ in Eq. (9), using Eq. (4), and putting $Y(\mathrm{Fo}) = T(1, \mathrm{Fo}) - T_c(\mathrm{Fo})$, we obtain an integral equation in $Y(\mathrm{Fo})$

$$Y(\mathrm{Fo}) = f(\mathrm{Fo}) + \int_0^{\mathrm{Fo}} K(\mathrm{Fo}, \omega)\, \mathrm{Bi}(\omega)\, Y(\omega)\, d\omega, \tag{10}$$

where

$$K(\mathrm{Fo}, \omega) = -\left\{1 + 2\sum_{n=1}^{\infty} \exp[-\mu_n^2 (\mathrm{Fo} - \omega)]\right\},$$

$$f(\mathrm{Fo}) = T_0 - T_c(\mathrm{Fo}).$$

From the last equation we obtain an integral equation for the heat flow through the surface $x = \delta$ in the form

$$q(\mathrm{Fo}) = F(\mathrm{Fo}) + \mathrm{Bi}(\mathrm{Fo}) \int_0^{\mathrm{Fo}} K(\mathrm{Fo}, \omega)\, q(\omega)\, d\omega, \tag{11}$$

$$F(\mathrm{Fo}) = f(\mathrm{Fo})\, \mathrm{Bi}(\mathrm{Fo}).$$

It should be noted, however, that in the case considered we have to deal with a Volterra integral equation of the second kind with a special type of kernel having a singularity along the line $\omega = \mathrm{Fo}$. Nevertheless, as is easily seen expressions

$$\int_0^{\mathrm{Fo}} K(\mathrm{Fo}, \omega)\, d\omega, \qquad \int_0^{\mathrm{Fo}} K^2(\mathrm{Fo}, \omega)\, d\omega$$

remain bounded.

The generally accepted methods [7] for the numerical solution of Eqs. (11) cannot be used. Below we consider an algorithm for the numerical solution of Eq. (11), in which the effect of the singularity is eliminated.

Our problem is to obtain a set of values of the heat flow as a function of different values of the Fourier numbers, namely,

$$q(\mathrm{Fo}^0) = q(0), \qquad q(\mathrm{Fo}^1) = q\left(\frac{ah}{\delta^2}\right), \; \ldots, \; q(\mathrm{Fo}^k) = q\left(\frac{akh}{\delta^2}\right).$$

Here h is a previously chosen (fairly small) time interval. Putting ah/δ^2, and substituting into Eq. (11) instead of the variable Fo its specific value s, Eq. (11) becomes

$$q(s) = F(s) - \mathrm{Bi}(s)\int_0^s q(\omega)\,d\omega - 2\mathrm{Bi}(s)\times\sum_{n=1}^{\infty}\exp(-\mu_n^2 s)\int_0^s \exp(\mu_n^2\omega)\,q(\omega)\,d\omega. \tag{12}$$

Similarly for the moment ks we have

$$q(ks) = F(ks) - \mathrm{Bi}(ks)\int_0^{ks} q(\omega)\,d\omega - 2\mathrm{Bi}(ks)\times\sum_{n=1}^{\infty}\exp(-\mu_n^2 ks)\int_0^{ks}\exp(\mu_n^2\omega)\,q(\omega)\,d\omega. \tag{13}$$

Replacement of the integral equations (12) and (13) by a set of algebraic equations using the usual quadrature formulas (the trapezium, Simpson's, etc.) does not lead to the desired result due to the singularity in the kernel. In view of this we propose to use the Cotes type quadrature formula with a weighting function [8].

We then have for Eq. (12)*

$$\begin{gathered}
q(s) = F(s) - \mathrm{Bi}(s)\left\{[A_0 q(0) + B_0 q(s)] - 2\sum_{n=1}^{\infty}\exp(\mu_n^2 s)\,[A_{1n}^1 q(0) + B_{1n}^1 q(s)]\right\},\\
A_0 = -s\int_0^1 (y-1)\,dy = \frac{s}{2},\\
B_0 = s\int_0^1 y\,dy = \frac{s}{2},\\
A_{1n}^1 = -s\int_0^1 \exp(\mu_n^2 sy)(y-1)\,dy = \frac{\exp(\mu_n^2 s)}{s\mu_n^4} - \frac{1}{\mu_n^2} - \frac{1}{s\mu_n^4},\\
B_{1n}^1 = s\int_0^1 \exp(\mu_n^2 sy)\,y\,dy = \frac{\exp(\mu_n^2 s)}{\mu_n^2} - \frac{\exp(\mu_n^2 s)}{s\mu_n^4} + \frac{1}{s\mu_n^4}.
\end{gathered} \tag{14}$$

Instead of Eq. (13), using the additivity of the integrals and employing the Cotes type quadrature formulas we have

$$\begin{gathered}
q(ks) = F(ks) - \mathrm{Bi}(ks)\left\{A_0 q(0) + B_0 q(s) + A_1 q(s) + B_1 q(2s) + \right.\\
+ \cdots + A_{k-1} q[(k-1)s] + B_{k-1} q(ks) - 2\mathrm{Bi}(ks)\times\\
\times\sum_{n=1}^{\infty}\exp(-\mu_n^2 ks)\,[A_n^1 q(0) + B_n^1 q(s) + A_n^2 q(s) + B_n^2 q(2s) +\\
+ \cdots + A_n^{k-1} q((k-1)s) + B_n^{k-1} q((k-1)s) +\\
\left. + A_n^k q((k-1)s) + B_n^k q(ks)]\right\},
\end{gathered} \tag{15}$$

where

$$A_m = B_m = \frac{s}{2}\quad (m = 1, 2, \ldots, k-1),$$

$$A_n^k = \exp[\mu_n^2 (k-1)s]\,A_{1n}^1,\quad B_n^k = \exp[\mu_n^2(k-1)s]\,B_{1n}^1.$$

*In view of the smallness of the time interval h it is sufficient to confine ourselves to two terms in the quadrature formula.

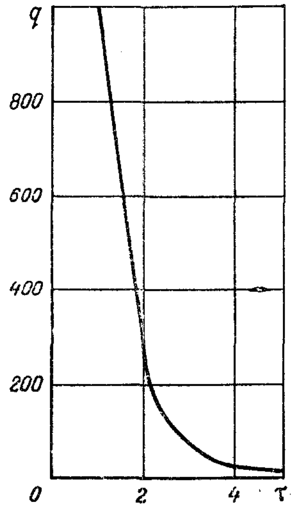

Fig. 1. Heat flow $Q(W/m^2)$ through the surface $x = \delta$ as a function of the time τ(s), obtained by computer calculation.

From the set of algebraic equations (14) and (15) we can find the values of the heat flow at the points ks (k = 1, 2, 3, . . .).

Equations (14) and (15) were solved numerically on a Minsk-32 computer from the following initial data: $\delta = 0.006$ m, $Q = 14.7 \times 10^{-8}$ m^2/sec, $\lambda = 0.245$ W/m·degree, $T_c = 150°C$, and $T_0 = 20°C$, with the variation of the Biot criterion being given by the relation

$$Bi = 0{,}91 \exp(-Fo).$$

Figure 1 shows the result of the numerical solution of the problem in the form of a curve of the heat flow through the external surface of the plate as a function of time.

NOTATION

T, temperature; Fo, Fourier number; a, thermal diffusivity; τ, time; ξ, dimensionless coordinate; x, space coordinate; Bi, Biot number; α (Fo), heat transfer coefficient; λ, thermal conductivity; T_0, initial plate temperature; T_c (Fo), external medium temperature; q, heat flux; δ, plate thickness; h, time pitch.

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